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On Macdonald's η-function Formula, the Laplacian and Generalized Exponents

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1. INTRODUCTION

1.1. Investigations involving the power series

$$\varphi(x) = \prod_{n=1}^{\infty} (1 - x^n)$$

have had a long history in mathematics. We will mention a few details. One interest arises since if we write $\varphi(x)^{-1} = \sum_{n=0}^{\infty} p(n)\, x^n$ then $n \mapsto p(n)$ is the classical partition function. The expansion $\varphi(x) = \sum_{n=-\infty}^{+\infty} (-1)^n\, x^{(3n^2+n)/2}$ is due to Euler. Arising out of his work on Theta functions, Jacobi (circa 1828) obtained the expansion (see, e.g. [4, p. 285])

$$\varphi(x)^3 = \sum_{n=0}^{\infty} (-1)^n\, (2n+1)\, x^{n(n+1)/2} \tag{1.1.1}$$

which such a well known combinatorist as Macmahon has called "the most remarkable formula in all of pure mathematics." (I thank D.N. Verma for this remark.) The function $\eta(x) = x^{1/24}\varphi(x)$ is called the Dedekind η-function. (More usually as a function of z in the upper half-plane where we substitute $x = e^{2\pi i z}$. It is then a modular function of z). The expansion of $\eta(x)^{24}$ has $\tau(n)$, the Ramanujan τ-function, as coefficients and an expression for this function has been given by F. Dyson (see [2, p. 636]). An expansion for $\varphi(x)^{10}$ is due to Winquist [11] enabling him to obtain a simple proof of Ramanujan's result that $p(11m + 6) \equiv 0 \bmod 11$. The numbers 3, 10 and 24 are the dimensions of the simple compact Lie groups $SU(2)$, Spin(5) and $SU(5)$. If K

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is any compact simply connected simple Lie group then I.G. Macdonald [7] has a remarkable formula for $\eta(x)^{\dim K}$ which generalizes the results stated above.

1.2. Let $l = \operatorname{rank} K$ and let $T \subseteq K$ be a maximal torus. Let $\mathbf{t}, \mathbf{k}$ be the Lie algebra of T, K and let $\mathbf{h}^*$ be the pure imaginary dual of $\mathbf{t}$. The character group $\hat{T}$ of T is isomorphic to $\mathbb{Z}^l$ and may be naturally identified with a lattice $Z \subseteq \mathbf{h}^* \cong \mathbb{R}^l$.

Let $L(\Sigma) \subset Z$ be the sublattice generated by the roots Σ of (T, K). Also let h be the Coxeter number of K. By definition, if W is the Weyl group of (T, K) and $\sigma \in W$ is the Coxeter element then $h = \text{order } \sigma$. For example if $K = SU(n)$ then $W = S_n$, the symmetric group on n-letters, and we can take $\sigma = (1, 2,..., n)$ (the permutation with 1 cycle). Thus $n = h$ for $K = SU(n)$. Next let Σ_+ be a system of positive roots and let (λ, β) be the bilinear form on $\mathbf{h}^*$ induced by the Killing form. Let $D = \{\lambda \in Z \mid (\lambda, \beta) \geqslant 0 \text{ for all } \beta \in \Sigma_+\}$ so that by the Cartan–Weyl theory D, as the highest weights, parametrizes $\hat{K}$, the set of equivalence classes of irreducible K-modules. In fact for each $\lambda \in D$ let

$$\pi_\lambda : K \to \operatorname{Aut} V_\lambda$$

be an irreducible representation of K with highest weight λ. Also let $\rho = \frac{1}{2} \sum_{\beta > 0} \beta$ and let for $\nu \in Z$

$$d(\nu) = \prod_{\beta>0} (\nu + \rho, \beta) \Big/ \prod_{\beta>0} (\rho, \beta)$$

so that by Weyl's formula $d(\lambda) = \dim V_\lambda$ for $\lambda \in D$. However for general $\nu \in Z$, $d(\nu)$ can be zero or negative. In case K is simply-laced ((β, β) is constant for $\beta \in \Sigma$) then Macdonald's formula may be written

$$\eta(x)^{\dim K} = \sum_{\nu \in hL(\Sigma)} d(\nu)\, x^{(\nu+\rho, \nu+\rho)}. \tag{1.2.1}$$

This formula for $\eta(x)^{\dim K}$ is in terms of a sum over a lattice. One senses here, however, that there is an underlying statement in terms of a sum over the irreducible representations of K. This is explicit in Jacobi's formula (1). In fact let $x_1, ..., x_d$ be an orthonormal base of $\mathbf{g} = \mathbf{k} + i\mathbf{k}$, so that in the universal enveloping algebra of $\mathbf{g}$, $\sum_{i=1} x_i^2 = z$ is the Casimir element. Thus $\pi_\lambda(z)$ is a scalar operator on V_λ and one knows the scalar $c(\lambda)$ is given by $c(\lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho)$. In (1.1.1) the exponents $n(n+1)/2 = c(\lambda)$ where π_λ is the $2n+1$ dimensional representation of $SU(2)$. But now in general

$$\varphi(x)^{\dim K} = \eta(x)^{\dim K}\, x^{-\dim K/24}.$$

However the "strange formula" of Freudenthal–de Vries (see [3, p. 243]) asserts that $\dim K/24 = (\rho, \rho)$. Thus by changing the summation from $hL(\Sigma)$ to D one can expect

$$\varphi(x)^{\dim K} = \sum_{\lambda \in D} \epsilon(\lambda) \dim V_\lambda\, x^{c(\lambda)}$$

where $\epsilon(\lambda)$ is some, yet to be determined, weighting of the representations $\{\pi_\lambda\} = \hat{K}$. For $SU(2)$ by Jacobi's formula the weighting is $\epsilon(\lambda) = 0$ if $\dim V_\lambda$ is even and alternating in sign for the odd dimensional representations. But this is exactly how the nontrivial element of the Weyl group operates on the zero weight space of these representations. Generalizing the usual action of W on $\mathbf{t}$, for any $\lambda \in D$ we obtain a representation of W

$$\theta_\lambda \colon W \to \operatorname{Aut} V_\lambda^T$$

on the space V_λ^T of T-invariants. This connection of representations of W and K, I think, is interesting. For example if $K = SU(n)$ then there is a natural set $\lambda_i \in D$, $i = 1, 2, \ldots, p(n)$, such that θ_{λ_i} is irreducible and runs over all the irreducible representations of $W = S_n$. Now if one looks at a character table for S_n it is a striking fact that one always has $\chi(\tau) \in \{1, 0, -1\}$ when $\tau = (1, 2, \ldots, n)$. A generalization of this and also a generalization of Jacobi's formula is given in

THEOREM 1. *If K is arbitrary then for any $\lambda \in D$ one always has* $\operatorname{tr} \theta_\lambda(\tau) = 1, 0, -1$ *where $\tau \in W$ is the Coxeter element. Moreover if K is simply laced then the weighting* $\epsilon(\lambda) = \operatorname{tr} \theta_\lambda(\tau)$. *That is*

$$\varphi(x)^{\dim K} = \sum_{\lambda \in D} \operatorname{tr} \theta_\lambda(\tau) \dim V_\lambda\, x^{c(\lambda)}.$$

1.3. We introduce two special kinds of elements in K. If $a \in K$ is regular (the centralizer of a in K is a torus) then the order of ad a is $\geqslant h$. We call an element $a \in K$ principal if (1) a is regular and (2) the order of Ad a equals h. It is then a fact that any two principal elements are conjugate.

Next an element $a \in K$ is called principal of type ρ if it is conjugate to the (regular) element $\exp 2x_\rho$ where $x_\rho \in \mathfrak{t}$ is that element such that $\langle \beta, x_\rho \rangle = 2\pi i(\beta, \rho)$ for any $\beta \in \Sigma$.

Remark 1. If K is simply-laced then the two notions are the same. That is, $a \in K$ is principal if and only if it is principal of type ρ.

Elements of the type introduced have remarkable character values.

THEOREM 2. *If $a \in K$ is principal then for any $\lambda \in D$ one has $\chi_\lambda(a) = 1$, -1 or 0, where χ_λ is the character of π_λ. In fact $\chi_\lambda(a) = \operatorname{tr} \theta_\lambda(\tau)$ where τ is the Coxeter element.*

Theorem 2 asserts that we may substitute $\chi_\lambda(a)$ for $\operatorname{tr} \theta_\lambda(\tau)$ in the formula (Theorem 1) for $\varphi(x)^{\dim K}$. But more than that is true. If we use a principal element of type ρ instead we do not have to assume that K is simply-laced in the formula for $\varphi(x)^{\dim K}$.

THEOREM 3. *Let $a \in K$ be a principal element of type ρ. Then $\chi_\lambda(a) = 1$, -1 or 0 for any $\lambda \in D$. Furthermore one has*

$$\varphi(x)^{\dim K} = \sum_{\lambda \in D} \chi_\lambda(a) \dim V_\lambda \, x^{c(\lambda)}.$$

1.4. If M is a compact Riemannian manifold and Δ is the Laplace–Beltrami operator on M there is a considerable amount of mathematical activity concerned with the question as to whether the complex valued function $s \mapsto \operatorname{tr}(\Delta + c)^{-s}$ defined on a half plane and extended meromorphically satisfies a functional equation. Here c is a constant >0. Consider the question when $M = K$. A natural choice for c from many points of view is (ρ, ρ). But now if $\operatorname{Re} s > \dim K/2$ then $(\Delta + (\rho, \rho))^{-s}$ is given by convolution by a function on K which we may identify with the operator $(\Delta + (\rho, \rho))^{-s}$. But then

$$\operatorname{tr}(\Delta + (\rho, \rho))^{-s} = (\Delta + (\rho, \rho))^{-s}(e)$$

where $e \in K$ is the identity. Our next statement is that if a principal element of type ρ is substituted for e then indeed the resulting function of s is holomorphic and satisfies a functional equation. In fact, since $\eta(e^{2\pi i z})$ is modular in z with zero constant term then the same is true for $\eta^{\dim K}$ and hence the Mellin transform $M(\eta^{\dim K})(s) = \Phi(s)$ is holomorphic and satisfies

$$\Phi\left(\frac{\dim K}{2} - s\right) = \Phi(s) \tag{1.4.1}$$

But now by the Peter–Weyl theorem one can sum the right side of the formula in Theorem 3 when $x^{c(\lambda)}$ is replaced by $(c(\lambda) + (\rho, \rho))^{-s}$. Thus one has

THEOREM 4. *Let $a \in K$ be a principal element of type ρ then*

$$M(\eta^{\dim K})(s) = (2\pi)^{-s}\,\Gamma(s)(\Delta + (\rho, \rho))^{-s}\,(a)$$

so that the function $s \mapsto (\Delta + (\rho, \rho))^{-s}(a)$ is everywhere holomorphic and satisfies a functional equation (K here is arbitrary).

If K is simply-laced there is another statement one can make which in effect applies to K/T rather than K. If the Poincaré polynomial of K is written

$$p_K(t) = \prod_{i=1}^{l} (1 + t^{2m_i+1}), \qquad m_1 \leqslant m_2 \leqslant \cdots \leqslant m_l\,,$$

then the integers m_i are called the exponents of K. More generally if $\lambda \in D$ and $l(\lambda) = \dim V_\lambda{}^T$ there is a naturally associated sequence of integers, $m_1(\lambda) \leqslant \cdots \leqslant m_{l(\lambda)}(\lambda)$, which are called the generalized exponents (see [6, p. 394]). The terminology is justified in that $m_i(\psi) = m_i$ where $\psi \in D$ is the highest root of Σ. We recall the definition. Let S denote the ring of complex-valued polynomial functions on $\mathbf{k}$ and let S_i denote the homogeneous component of S of degree i. The adjoint action of K on $\mathbf{k}$ induces an action of K on S and let $J = S^K$ be the ring of polynomial invariants. On the other hand let $H \subseteq S$ denote the set of harmonic polynomials in the generalized sense i.e. all $f \in S$ such that $\partial f = 0$ where ∂ is any constant coefficient differential operator without constant term which commutes with the action of K. One knows that $S = J \otimes H$ and that H is a K-submodule with finite multiplicities. In fact the multiplicity of π_λ is $l(\lambda)$. Indeed if $H_\lambda \subseteq H$, for $\lambda \in D$, is the primary π_λ-component then the $m_i(\lambda)$ are defined in that we can write

$$H_\lambda = \bigoplus_{i=1}^{l(\lambda)} H_\lambda{}^i$$

where $H_\lambda{}^i \subseteq S_{m_i(\lambda)}$ and $H_\lambda{}^i$ is K-irreducible.

Now let $\omega = e^{2\pi i/h}$. By results of Coxeter and Coleman one knows that the eigenvalues of $\theta_\psi(\tau)$ are ω^{m_i} where $\tau \in W$ is the Coxeter element. More generally we have proved (see [5, p. 399]) that the eigenvalues of $\theta_\lambda(\tau)$ are $\omega^{m_i(\lambda)}$, $i = 1,..., l(\lambda)$, for any $\lambda \in D$. Thus one has for any $\lambda \in D$

$$\operatorname{tr} \theta_\lambda(\tau) = \omega^{m_i(\lambda)} + \cdots + \omega^{m_{l(\lambda)}(\lambda)} = \begin{cases} 1 \\ -1 \\ 0 \end{cases} \tag{1.4.2}$$

Now let Ω be the operator on H defined so that Ω is multiplication by ω^i on $H_i = H \cap S_i$. Now let $\square$ be the operator on H defined by the restriction of the Casimir operator to H (via the adjoint action). This corresponds to the Laplace–Beltrami operator on K/T (using the Killing form to define the metric) since H as a K-module is isomorphic to the space of K-finite functions on K/T. If we substitute (1.4.2) in Theorem 1, replace φ by η and recall that $\bigoplus_\lambda H_\lambda = H$ then one has (convergence is easy for $\operatorname{Re} s > 1 + \dim K$.

THEOREM 5. *Assume K is simply-laced and let $M(\eta^{\dim K})(s)$ be the Mellin transform of the Dedekind η-function to the* dim Kth *power. Then*

$$M(\eta^{\dim K})(s) = (2\pi)^{-s}\, \Gamma(s) \operatorname{tr} \Omega(\square + (\rho, \rho))^{-s}.$$

1.5. I wish to thank R. Godement for conversations about modular forms and Hecke theory.

2. THE η-FUNCTION FORMULAS OF MACDONALD

2.1. Let K be any simply connected compact simple Lie group and let $T \subseteq K$ be a maximal torus. Let $d = \dim K$ and $l = \dim T = \operatorname{rank} K$. Let $\mathbf{t}$ and $\mathbf{k}$ be the Lie algebras of T and K and let $\mathbf{h}$ and $\mathbf{g}$ be their respective complexifications. Let Σ be the set of roots for the pair $(\mathbf{h}, \mathbf{g})$ so that $\Sigma \subseteq \mathbf{h}^*$ where $\mathbf{h}^* = \operatorname{Hom}_{\mathbb{R}}(\mathbf{t}, i\mathbb{R})$ is the imaginary dual to $\mathbf{t}$. Thus $\mathbf{h}^* \simeq \mathbb{R}^l$ and if $L(\Sigma)$ is the $\mathbb{Z}$-space of Σ then $L(\Sigma) \simeq \mathbb{Z}^l$ is a lattice in $\mathbf{h}^*$.

The Killing form on $\mathbf{g}$ induces a bilinear form on $\mathbf{h}^*$ whose value on $\mu, \nu \in \mathbf{h}^*$ is denoted by (μ, ν). This form on $\mathbf{h}^*$ is positive definite defining a norm $|\mu| = (\mu, \mu)^{1/2}$ on $\mathbf{h}^*$. One says that K is simply-laced (type A, D or E) in case all the roots have the same length.

One defines a lattice $N \subseteq \mathbf{h}^*$, containing $L(\Sigma)$, by putting

$$N = \left\{ \nu \in \mathbf{h}^* \;\middle|\; \frac{2(\nu, \beta)}{(\beta, \beta)} \in \mathbb{Z} \text{ for all } \beta \in \Sigma \right\}.$$

The elements of N are called the integral linear forms on $\mathbf{t}$. One knows that N parameterizes the character group $\hat{T}$ of T. In fact if $\nu \in N$ there exists a unique well-defined character e^ν of T so that if $a \in T$, $a = \exp x$, $x \in \mathbf{t}$ then $e^\nu(a) = e^{\langle \nu, x \rangle}$. Moreover the map $N \to \hat{T}$, $\nu \mapsto e^\nu$ is a bijection.

Let $\Sigma_+ \subseteq \Sigma$ be a fixed system of positive roots and $D = \{\lambda \in N \mid (\lambda, \beta) \geqslant 0 \text{ for all } \beta \in \Sigma_+\}$. By the Cartan–Weyl theory one knows that D parameterizes $\hat{K}$, the set of equivalence classes of finite dimensional irreducible representations of K (or $\mathbf{k}$). That is, if $\lambda \in D$, there exists, up to equivalence, a unique irreducible representation $\pi_\lambda \colon K \to \operatorname{Aut} V_\lambda$ such that λ is the highest weight of π_λ. Moreover the map $D \to \hat{K}$, $\lambda \to \{\pi_\lambda\}$ is a bijection.

2.2. Now let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple positive roots. For any $\beta \in \Sigma$ let $n_i(\beta) \in \mathbb{Z}$ be defined so that $\beta = \sum_{i=1}^l n_i(\beta)\alpha_i$. One puts $O(\beta) = \sum_{i=1}^l n_i(\beta)$. An important number in the paper is the Coxeter number h. This may be defined in many ways. We will define h here by putting

$$h = O(\psi) + 1 \tag{2.2.1}$$

when $\psi \in \Sigma_+$ is the highest root. (That is, ψ is the highest weight of the adjoint representation, i.e., $\pi_\psi \cong \mathrm{Ad}$). Throughout this paper ψ will denote the highest root.

Now let $x_i \in \mathbf{g}$, $i = 1, 2, \ldots, d$ be a basis of $\mathbf{g}$ and let $g_{ij} = \operatorname{tr} ad x_i \, ad x_j$. Then the $d \times d$ matrix g_{ij} is invertible and let g^{ij} be its inverse. Then the element $z = \sum_{i,j} g^{ij} x_i x_j$ in the universal enveloping algebra U of $\mathbf{g}$ is called the Casimir element. For any $\lambda \in D$ the representation π_λ extends to U and since $z \in \text{Center } U$ the operator $\pi_\lambda(z)$ is a scalar, $c(\lambda)$, multiple of the identity. One knows that

$$c(\lambda) = |\lambda + \rho|^2 - |\rho|^2 \tag{2.2.2}$$

where $\rho = \frac{1}{2} \sum_{\beta \in \Sigma_+} \beta$. See e.g. [10] and note that Ω in [10] equals $-z$ here.

Now by definition of z one clearly has $c(\psi) = 1$. Thus (2.2.1) implies

$$1 = 2(\rho, \psi) + (\psi, \psi) \tag{2.2.3}$$

and hence if we put $h_P = 1/(\psi, \psi)$ one has

$$h_P = \frac{2(\rho, \psi)}{(\psi, \psi)} + 1 \tag{2.2.4}$$

Furthermore one has $h_P \in \mathbb{Z}$ since one knows $\rho \in D \subseteq N$.

PROPOSITION 2.2. *If K is simply-laced one has $h_P = h$.*

Proof. In any case one knows that

$$\frac{2(\rho, \alpha_i)}{(\alpha_i, \alpha_i)} = 1 \tag{2.2.5}$$

$i = 1, 2,..., l$. But then if K is simply-laced one has $2(\rho, \alpha_i)/(\psi, \psi) = 1$. Since $\psi = \sum_{i=1}^{l} n_i(\psi)\alpha_i$ this implies that $2(\rho, \psi)/(\psi, \psi) = O(\psi)$ and hence the result follows from (2.2.1) and (2.2.3). Q.E.D.

2.3. Now let

$$\varphi(x) = \prod_{n=1}^{\infty} (1 - x^n) \tag{2.3.1}$$

so that $x^{1/24}\varphi(x) = \eta(x)$ is the Dedekind η-function. By specializing Theorem 8.7 in [7] at the identity Macdonald obtains two formulas involving $\eta(x)$ and K. These two formulas and certain notation associated with them will be indexed throughout this paper, respectively, by the letters P and Q. In order to write down these two formulas we will have to define two sublattices M_P and M_Q of $L(\Sigma)$.

If K is simply-laced the two formulas are the same. In particular we will have $M_P = M_Q$ in this case.

By definition M_P is the $\mathbb{Z}$-span of the set $\{\beta/(\beta, \beta) \mid \beta \in \Sigma\}$.

PROPOSITION 2.3.1. *For any root β one has $1/(\beta, \beta) \in \mathbb{Z}$ so that M_P is sublattice of $L(\Sigma)$.*

To prove Proposition 2.3.1 we first recall that if, for any root β we put $m(\beta) = (\psi, \psi)/(\beta, \beta)$ then $m(\beta)$ is an integer. In fact

$$m(\beta) = 1, 2, \text{ or } 3. \tag{2.3.2}$$

The statement (2.3.2) is well known from the classification of root systems. Without appeal to classification to prove (2.3.2) we may assume that β is not linearly dependent on ψ and that $(\psi, \beta) \neq 0$. The last assumption is justified in that otherwise β may be replaced by a Weyl group translate of β. The irreducibility of Ad implies that not all the elements in the Weyl group orbit of β can be orthogonal to ψ. Now

(2.3.2) is an easy consequence of the following facts (1) $(\psi, \psi) \geqslant (\beta, \beta)$, which follows since ψ is the highest weight of Ad, (2) $2(\psi, \beta)/(\beta, \beta)$ and $2(\psi, \beta)/(\psi, \psi)$ are integers and (3) $1 \leqslant 2(\psi, \beta)/(\beta, \beta) \cdot 2(\psi, \beta)/(\psi, \psi) \leqslant 3$ by the Schwartz inequality. But then (1), (2) and (3) imply that $2\,|(\psi, \beta)|/(\psi, \psi) = 1$ and hence $2\,|(\psi, \beta)|/(\beta, \beta) = (\psi, \psi)/(\beta, \beta) = 1$, 2, or 3.

Proof of Proposition 2.3.1. We recall (see (2.2.4)) that $h_P = 1/(\psi, \psi) \in \mathbb{Z}$. But then by (2.3.2)

$$m(\beta)\, h_P = \frac{1}{(\beta, \beta)} \in \mathbb{Z}. \qquad \text{Q.E.D.}$$

The other sublattice $M_Q \subseteq L(\Sigma)$ is defined by

$$M_Q = hL(\Sigma).$$

Remark 2.3. It is clear that the elements $h\alpha_i$, $i = 1, 2, \ldots, l$ are a $\mathbb{Z}$-basis of M_Q. On the other hand one knows that the elements $\alpha_i/(\alpha_i, \alpha_i)$, $i = 1, 2, \ldots, l$ are a $\mathbb{Z}$-basis of M_P. To prove that it suffices to show that $\beta/(\beta, \beta)$ is in the $\mathbb{Z}$-span of the $\alpha_i/(\alpha_i, \alpha_i)$ for any $\beta \in \Sigma_+$. This is proved by induction on $O(\beta)$. If β is not simple there exists a simple root α_j such that $(\alpha_j, \beta) > 0$. But then if $\gamma \in \Sigma_+$ is the image of β with respect to the reflection defined by α_j one has $\gamma = \beta - 2(\beta, \alpha_j)\alpha_j/(\alpha_j, \alpha_j)$ so that $O(\gamma) < O(\beta)$ and hence $\gamma/(\gamma, \gamma)$ is in the $\mathbb{Z}$-span of the $\alpha_i/(\alpha_i, \alpha_i)$ by induction. However $(\gamma, \gamma) = (\beta, \beta)$ so that

$$\frac{\beta}{(\beta, \beta)} = \frac{\gamma}{(\gamma, \gamma)} + \frac{2(\beta, \alpha_j)}{(\beta, \beta)} \cdot \frac{\alpha_j}{(\alpha_j, \alpha_j)}$$

proving that $\beta/(\beta, \beta)$ is in the $\mathbb{Z}$-span of the $\alpha_i/(\alpha_i, \alpha_i)$.

PROPOSITION 2.3.2. *$M_P = M_Q$ if K is simply-laced.*

Proof. If K is simply-laced then for any root β one has $1/(\beta, \beta) = 1/(\psi, \psi) = h_P$. But $h_P = h$ by Proposition 2.2. Q.E.D.

2.4. Now let $\nu \mapsto d(\nu)$ be the function on N defined by putting $d(\nu) = \prod_{\beta \in \Sigma_+} (\nu + \rho, \beta)/\prod_{\beta \in \Sigma_+} (\rho, \beta)$ so that by Weyl's formula one has

$$d(\lambda) = \dim V_\lambda \qquad \text{for} \quad \lambda \in D.$$

Macdonald's two formulas are stated in Theorem 2.4. Recall that $d = \dim K$.

THEOREM 2.4. *As a formal power series in x one has*

$$(P) \qquad \eta(x)^d = \sum_{\nu\in M_P} d(\nu)\, x^{(\nu+\rho,\nu+\rho)}$$

and also

$$(Q) \qquad \left(\prod_{i=1}^{l} \eta(x^{h(\alpha_i,\alpha_i)})\right)^{h+1} = \sum_{\nu\in M_Q} d(\nu)\, x^{(\nu+\rho,\nu+\rho)}.$$

Remark 2.4. The notation in Theorem 2.4 is of course different from that used in [7]. To see that Theorem 2.4 is in fact stated in [7] consider first equation (P). We note that in the notation of [7] if the bilinear form $\langle u, v\rangle$ is taken to be $\Phi_R(u, v)$ then $g = \frac{1}{2}$ (see Proposition 7.13 in [7]) but then M in [7] is just M_P here so that (P) here is just 8.9 in [7]. Next note that (Q) here is 8.13 in [7]. To see that, replace, in 8.13, $X^{h^{-1}}$ by x and $h\lambda$ by $\nu \in M_Q$. The power $h + 1$ on the right side of 8.13 has, by misprint, been omitted.

3. TWO DISTINGUISHED REGULAR ELEMENTS IN K

3.1. Since K is simply-connected the notion of regularity is easier to deal with then would otherwise be the case. That is, since K is simply-connected one knows that for any $a \in K$, the centralizer K^a, of a, in K, is always connected. An element $a \in K$ is called regular in case K^a is a torus (and hence clearly a maximal torus).

Now let $x_P \in \mathbf{t}$ be defined so that $\langle \nu, x_P\rangle = 2\pi i(\nu, 2\rho)$ for all $\nu \in \mathbf{h}^*$. In particular note that if α is a simple root then

$$\langle \alpha, x_P\rangle = 2\pi i(\alpha, \alpha). \tag{3.1.1}$$

Now define $a_P \in T$ by putting $a_P = \exp x_P$. Next define $x_Q \in \mathbf{t}$ so that $\langle \beta, x_Q\rangle = 2\pi i\, O(\beta)/h$ for any $\beta \in \Sigma$. In particular note in this case that if α is a simple root one has

$$\langle \alpha, x_Q\rangle = 2\pi i \frac{1}{h}. \tag{3.1.2}$$

Define $a_Q \in T$ by putting $a_Q = \exp x_Q$.

PROPOSITION 3.1. *One has $a_P = a_Q$ if and only if K is simply-laced.*

Proof. This is immediate from (3.1.1), (3.1.2) and Proposition 2.2. Q.E.D.

Now for any $\lambda \in D$ let χ_λ be the character of the representation π_λ. Thus $\chi_\lambda(a) = \operatorname{tr} \pi_\lambda(a)$ for $a \in K$. The theorem we wish to prove here is the following one which says that, over all $\lambda \in D$, $\chi_\lambda(a_P)$ and $\chi_\lambda(a_Q)$ remarkably take only the values 1, -1 or 0 and that Macdonald's formulas are directly expressed in terms of these character values.

THEOREM 3.1. *The elements a_P and a_Q in T are regular. Furthermore for any $\lambda \in D$ the character values $\chi_\lambda(a_P)$ and $\chi_\lambda(a_Q)$ are either* 1, -1 *or* 0 *and one has*

$$\eta(x)^d = \sum_{\lambda \in D} \chi_\lambda(a_P) \dim V_\lambda \, x^{(\lambda+\rho, \lambda+\rho)}$$

and

$$\left(\prod_{i=1}^{l} \eta(x^{h(\alpha_i, \alpha_i)})\right)^{h+1} = \sum_{\lambda \in D} \chi_\lambda(a_Q) \dim V_\lambda \, x^{(\lambda+\rho, \lambda+\rho)}.$$

Theorem 3.1 will be proved in Section 3.6.

Remark 3.1. For the special case where $\lambda \in hL(\Sigma)$ it is immediate from Weyl's character formula that $\chi_\lambda(a_Q) = 1$. This has been noted by Macdonald in [7] and he has shown that this fact leads to Theorem 8.16 in [7].

3.2. To detect regularity for an element $a \in K$ one has only to consider the invariants of $\operatorname{Ad} a$. For $a \in T$ one clearly has

PROPOSITION 3.2.1. *Let $a \in T$. Then a is regular if and only if $e^\beta(a) \neq 1$ for all $\beta \in \Sigma$.*

Questions of regularity or conjugacy in T are illuminated by introduction of Weyl's fundamental simplex. Let $y_j \in \mathbf{h}$, $j = 1, 2, \ldots, l$ be defined so that $\langle \alpha_i, y_j \rangle = \delta_{ij}$. The set

$$F = \left\{ x \in \mathbf{t} \;\middle|\; x = 2\pi i \sum_{j=1}^{l} \frac{r_j y_j}{n_j(\psi)}, 0 \leqslant r_j, \sum_{j=1}^{l} r_j \leqslant 1 \right\}$$

is called Weyl's fundamental simplex. One knows that given any $a \in K$

there exists a unique element $x(a) = 2\pi i \sum_{j=1}^{l} (r_j(a)/n_j(\psi))\, y_j \in F$ such that a is conjugate to $\exp x(a)$. It then follows easily that a is regular if and only if $x(a)$ is in the interior of F, i.e., $0 < r_j(a)$ for all j and $\sum_{j=1}^{l} r_j(a) < 1$.

Let W be the Weyl group of (T, K) regarded, as usual, as operating on $\mathbf{h}$, $\mathbf{h}^*$ and T. Also let W_A be the affine Weyl group regarded as operating on $\mathbf{t}$. The group W_A can be described in two ways (1) it is the group generated by the reflections in the $l + 1$ walls of F or (2) W_A is the semidirect product of W with the group of translations in $\mathbf{t}$ by all $x \in \mathbf{t}$ such that $\exp x = e$. The second description of W_A implies

PROPOSITION 3.2.2. *If $x, y \in \mathbf{t}$ then* $\exp x$ *and* $\exp y$ *are conjugate in K if and only if x and y are W_A-conjugate.*

This statement together with the uniqueness of $x(a)$ defined above imply that F is a fundamental domain for the action of W_A on $\mathbf{t}$. This however together with the first description of W_A implies that if $x \in \mathbf{t}$ then $\exp x$ is regular if and only if $\tau x = x$ for $\tau \in W_A$, implies τ is the identity. But then this implies the familiar

PROPOSITION 3.2.3. *If $a \in T$ then a is regular if and only if $\sigma a = a$ for $\sigma \in W$ implies σ is the identity.*

3.3. Let R denote either P or Q. Since N is isomorphic to the character group $\hat{T}$ of T and since M_R is a sublattice of N one defines a finite subgroup Γ_R of T by putting

$$\Gamma_R = \{a \in T \mid e^\nu(a) = 1 \text{ for all } \nu \in M_R\}.$$

In fact the isomorphism $N \to \hat{T}$, $\nu \to e^\nu$ induces an isomorphism

$$N/M_R \cong \hat{\Gamma}_R \tag{3.3.1}$$

when $\hat{\Gamma}_R$ is the character group of Γ_R so that if the absolute sign denotes the order of the group one has

$$|N/M_R| = |\Gamma_R|. \tag{3.3.2}$$

Now by its definition it is clear that M_R is stable under the action of the Weyl group W. It follows therefore that the finite group Γ_R is also stable under the action of W. In particular then Γ_R is a finite union of W-orbits. The following result, which plays a key role in the proof of Theorem 3.1, asserts that there is exactly one W-orbit in Γ_R which has a trivial isotropy subgroup. That is, recalling Proposition 3.2.3, there is exactly one W-orbit in Γ_R which consists of regular elements.

LEMMA 3.3. *If $R = P$ or Q one has $a_R \in \Gamma_R$. Moreover an element $a \in \Gamma_R$ is regular if and only if it is of the form σa_R for some $\sigma \in W$. The element σ is necessarily unique.*

Proof. Consider first the case where $R = P$. Since Γ_P is stable under the action of W to find the regular elements in Γ_P it is enough to consider elements $a \in \Gamma_P$ of the form $a = \exp x$ where x is in the fundamental simplex F. Let $a \in \Gamma_P \cap \exp F$ so that $a = \exp x$ where $x = 2\pi i \sum_{j=1}^{l} (r_j(a)/n_j(\psi))\, y_j$ and $\langle \alpha_i, y_j \rangle = \delta_{ij}$ and (1) $r_j(a) \geqslant 0$, (2) $\sum_{j=1}^{l} r_j(a) \leqslant 1$. Now by Remark 2.3 the condition that $a \in \Gamma_P$ is exactly the condition that $\langle \alpha_j, x \rangle/(\alpha_j, \alpha_j) \in 2\pi i\, \mathbb{Z}$ for $j = 1, 2,..., l$. Thus one has $(r_j(a)/n_j(\psi)(\alpha_j, \alpha_j)) = k_j(a) \in \mathbb{Z}_+$ for $j = 1,..., l$. But clearly we can also invert this statement. This proves that the map $\Gamma_P \cap \exp F \to \mathbb{R}^l$, $a \mapsto (r_1(a),..., r_l(a))$ sets up a bijection between $\Gamma_P \cap \exp F$ and the set of all $(r_1,..., r_l) \in \mathbb{R}^l$ such that (1) $r_j \geqslant 0$, (2) $\sum r_j \leqslant 1$ and (3) $(r_j/n_j(\psi)(\alpha_j, \alpha_j)) = k_j \in \mathbb{Z}_+$. Solving for r_j this may be expressed more simply by saying that $\Gamma_P \cap \exp F$ is in 1–1 correspondence with the set of all l-tuples $(k_1,..., k_l)$ where $k_j \in \mathbb{Z}_+$ and $1 \geqslant \sum_{j=1}^{l} (\alpha_j, \alpha_j)\, n_j(\psi) k_j$. But then clearly the regular elements in $\Gamma_P \cap \exp F$ are in 1–1 correspondence with the set of all l-tuples of *positive* integers $(k_1,..., k_l)$ such that

$$1 > \sum_{j=1}^{l} (\alpha_j, \alpha_j)\, n_j(\psi)\, k_j. \tag{3.3.3}$$

However writing $\psi = \sum_{j=1}^{l} n_j(\psi)\alpha_j$ it follows from (2.2.5) that

$$2(\rho, \psi) = \sum_{j=1}^{l} (\alpha_j, \alpha_j)\, n_j(\psi). \tag{3.3.4}$$

But by (2.2.3) one has $2(\rho, \psi) = 1 - (\psi, \psi)$ so that $1 > 2(\rho, \psi)$. It follows therefore that $k_j = 1$ for $j = 1, 2,..., l$ is a solution to (3.3.3). That is, there exists a regular element $b \in \Gamma_P \cap \exp F$ such that $k_j(b) = 1$ for all j. But now $r_j(b) = (\alpha_j, \alpha_j)\, n_j(\psi)$ so that $(r_j(b)/n_j(\psi)) = (\alpha_j, \alpha_j)$. Thus $b = \exp y$ where $y = 2\pi i \sum_{j=1}^{l} (\alpha_j, \alpha_j)\, y_j$. But then $\langle \alpha_j, y \rangle = 2\pi i(\alpha_j, \alpha_j)$ and hence $y = x_P$ by (3.1.1). Thus $b = a_P \in \Gamma_P$. Also this proves that a_P is regular. To prove the lemma for the case where $R = P$ it suffices to prove that $k_j = 1$ is the only solution to (3.3.3). Indeed this follows from the fact that any element in Γ_P is W-conjugate to an element in $\Gamma_P \cap \exp F$.

Assume that $(k_1, ..., k_l)$ is a solution to (3.3.3) where $k_j \geqslant 1$ for all j and $k_i \geqslant 2$ for some i. But then

$$\sum_{j=1}^{l} (\alpha_j, \alpha_j)\, n_j(\psi)\, k_j \geqslant \sum_{j=1}^{l} (\alpha_j, \alpha_j)\, n_j(\psi) + (\alpha_i, \alpha_i)\, n_i(\psi) \tag{3.3.5}$$

$$\geqslant 2(\rho, \psi) + (\alpha_i, \alpha_i)\, n_i(\psi).$$

But now

$$\frac{\psi}{(\psi, \psi)} = \sum_{j=1}^{l} \left(\frac{n_j(\psi)(\alpha_j, \alpha_j)}{(\psi, \psi)}\right) \frac{\alpha_j}{(\alpha_j, \alpha_j)}.$$

However, by Remark 2.3, $\psi/(\psi, \psi)$ is in the $\mathbb{Z}$-span of the $\alpha_j/(\alpha_j, \alpha_j)$. It follows therefore that $n_j(\psi)(\alpha_j, \alpha_j)/(\psi, \psi) \in \mathbb{Z}_+$ and hence, for all j, $n_j(\psi)(\alpha_j, \alpha_j)/(\psi, \psi) \geqslant 1$. Thus $(\alpha_i, \alpha_i)\, n_i(\psi) \geqslant (\psi, \psi)$. But then by (3.3.5) one has

$$\sum_{j=1}^{l} (\alpha_j, \alpha_j)\, n_j(\psi)\, k_j \geqslant 2(\rho, \psi) + (\psi, \psi)$$

$$\geqslant 1$$

by (2.2.3). This contradicts (3.3.3). Thus $(1, ..., 1)$ is the unique positive solution to (3.3.3) proving the lemma for the case where $R = P$.

Now assume $R = Q$. If $a \in \exp F$ then as above, $a = \exp x$ where $x = 2\pi i \cdot \sum_{j=1}^{l} (r_j(a)/n_j(\psi))\, y_j$. But now by Remark 2.3 the elements $h\alpha_j$, $j = 1, 2, ..., l$ are a $\mathbb{Z}$-basis of M_Q. Thus $a \in \Gamma_Q \cap \exp F$ if and only if $\langle h\alpha_j, x\rangle \in 2\pi i\, \mathbb{Z}$ or if and only if $(r_j(a)/n_j(\psi))h = q_j(a) \in \mathbb{Z}_+$ for all j. Thus proceeding in a way similar to that for the case of Γ_P one obtains a bijection of $\Gamma_Q \cap \exp F$ with the set of all l-tuples $(q_1, ..., q_l)$ where $q_i \in \mathbb{Z}_+$ such that $\sum_{j=1}^{l} n_j(\psi) q_j/h \leqslant 1$. In this correspondence the regular elements in $\Gamma_Q \cap \exp F$ correspond to all l-tuples $(q_1, ..., q_l)$ where the q_i are positive integers and

$$\frac{\sum_j n_j(\psi)\, q_j}{h} < 1. \tag{3.3.6}$$

We will show that $q_j = 1$, for $j = 1, 2, ..., l$ is the unique solution. Indeed this is a solution because $(\sum_{j=1}^{l} n_j(\psi)/h) = O(\psi)/h < 1$ since, by definition $h = O(\psi) + 1$. If b is the regular element in $\Gamma_Q \cap \exp F$ such that $q_j(b) = 1$. Then $r_j(b) = n_j(\psi)/h$ so that $b = \exp y$ where $y = 2\pi i \sum_{j=1}^{l} (y_j/h)$. But then $\langle \alpha_j, y\rangle = 2\pi i(1/h)$. Thus $y = x_Q$ by (3.1.2) and hence $b = a_Q$. Thus we have proved that $a_Q \in \Gamma_Q$ and that a_Q is regular. We have only to show that $q_j = 1$ is the only positive integral solution to (3.3.6). Indeed assume $(q_1, ..., q_l)$ is another solution to (3.3.6) when $q_j \geqslant 1$ and $q_i \geqslant 2$ for some i. Then

$$\frac{\sum n_j(\psi)\, q_j}{h} \geqslant \frac{\sum n_j(\psi)}{h} + \frac{n_i(\psi)}{h}$$

$$= \frac{O(\psi) + n_i(\psi)}{h}.$$

But one knows that $n_j(\psi) \geqslant 1$ for all j (since ψ is a highest weight). Thus $(\sum n_j(\psi)q_j)/h \geqslant (O(\psi) + 1)/h = 1$ by definition of h. But this is a contradiction of (3.3.6), proving the lemma. Q.E.D.

3.4. Now by definition an element $a \in T$ is in Γ_Q in case $e^{h\beta}(a) = 1$ for all $\beta \in \Sigma$. That is, in much simpler terms,

$$\Gamma_Q = \{a \in T \mid (\operatorname{ad} a)^h = 1\}. \tag{3.4.1}$$

Now in [5] we defined the notion of principal element in Ad $\mathbf{g}$. See Section 6.7 and Section 5.2 in [5]. These elements form a distinguished conjugacy class. One of the theorems proved about principal elements in Ad $\mathbf{g}$ is Corollary 8.6 in [5]. This result asserts that if $b \in \operatorname{Ad} \mathbf{g}$ is regular (i.e., the invariant set of $\mathbf{g}^b$ is a Cartan subalgebra) and if k is the order of b then $k \geqslant h$ and $k = h$ if and only if b is principal (Note s in [5] equals h. See Section 8.3 in [5]). Here we are dealing with a compact simply-connected group K. Having to deal now with the center of K still does not destroy the conjugacy property.

THEOREM 3.4. *Let $a \in K$ be regular and let k be the order of* Ad a *(most likely ∞). Then $k \geqslant h$. Furthermore the set $C = \{a \in K \mid a$ regular and* $(\operatorname{Ad} a)^h = 1\}$ *is a single conjugacy class in K.*

Proof. The first statement follows from Corollary 8.6 in [5]. On the other hand by Lemma 3.3 and (3.4.1) one has $a_Q \in C$. But then any conjugate of a_Q lies in C. Conversely if $b \in C$ let $a \in T$ be a conjugate of b. But then $a \in \Gamma_Q$ by (3.4.1). But then a is conjugate to a_Q by Lemma 3.3. Q.E.D.

The elements in the conjugacy class C will be called the principal elements of K. In particular a_Q is a principal element. Principal elements in K are closely related to the Coxeter elements in the Weyl group. We recall that an element $\tau \in W$ is called a Coxeter element (called a Coxeter–Killing element in [5]) in case it is conjugate to the element $\sigma_1 \cdots \sigma_l \in W$ where σ_i is the reflection defined by the simple root α_i.

One of course knows that h is the order of a Coxeter element (See Theorem 8.4 in [5]). The relation to principal elements is closer than that. Consider the exact sequence.

$$1 \to T \to N(T) \to W \to 1 \tag{3.4.2}$$

where $N(T)$ is the normalizer of the torus T. Let $\gamma\colon N(T) \to W$ be the coset projection.

PROPOSITION 3.4.1. *Let $\tau \in W$ be any Coxeter element and let $a \in N(T)$ be arbitrary such that $\gamma(a) = \tau$. Then a is a principal element in K.*

Proof. If $a \in K$ it is clear from Corollary 8.6 in [5] that a is a principal element in K if and only if $\operatorname{Ad} a$ is a principal element in $\operatorname{Ad} \mathfrak{g}$, in the terminology of [5]. The result then follows from Theorem 8.6 in [5] which asserts that $\operatorname{Ad} a$ is a principal element in $\operatorname{Ad} \mathfrak{g}$. Q.E.D.

We shall make use of the following result of Steinberg.

LEMMA 3.4.1. *Let $\tau \in W$ be a Coxeter element. Then*

$$\det(1 - \tau) = |\operatorname{Cent} K|.$$

Proof. Since the non-real eigenvalues of τ occur in conjugate pairs evidently $\det(1 - \tau) \geqslant 0$. Thus it is enough only to show $|\det(1 - \tau)| = |\operatorname{cent} K|$. Now one knows that the set of simple roots Π may be written as a disjoint union $\Pi_1 \cup \Pi_2$ where all the roots in Π_i, $i = 1, 2$, are orthogonal to one another. Thus if τ_i is the product of the reflections corresponding to the roots in Π_i then τ_i is of order 2. On the other hand $\tau_1\tau_2$ is a Coxeter element and hence we can take $\tau = \tau_1\tau_2$. (See [9] for the details). Thus

$$|\det(1 - \tau)| = |\det(\tau_1 - \tau_2)|. \tag{3.4.3}$$

On the other hand since $\operatorname{Cent} K$ is the kernel of the adjoint representation and $\operatorname{Cent} K \subseteq T$ it follows that the isomorphism $N \to \hat{T}$,

$\nu \mapsto e^\nu$, defines an isomorphism of $N/L(\Sigma)$ with the character group of Cent K. Thus

$$|N/L(\Sigma)| = |\operatorname{Cent} K|. \tag{3.4.4}$$

If $\lambda_i \in \mathbf{h}^*$, $i = 1,..., l$, are defined by $2(\lambda_l, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ so that the λ_i are the highest weights of the fundamental representations of K then one knows that the λ_i are a $\mathbb{Z}$-basis of N. The Cartan $l \times l$ matrix M, where $M_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$, is the just the matrix which expresses the α_j in terms of the λ_i so that by (3.4.4) one has

$$|\det M| = |\operatorname{Cent} K|. \tag{3.4.5}$$

But it is straightforward to observe that the matrix for $\tau_1 - \tau_2$ with respect to the basis Π is just the Cartan matrix M except for the signs of certain of the rows. But this implies $|\det(\tau_1 - \tau_2)| = |\det M|$. The result then follows from (3.4.3) and (3.4.5). Q.E.D.

Now let $m_1 \leqslant \cdots \leqslant m_l$ be the exponents of K. That is, the Poincaré polynomial, $p_K(t)$, of K equals $\prod_{i=1}^{l} (1 + t^{2m_i+1})$. Also let $\omega = e^{2\pi i/h}$. Now if $\tau \in W$ is a Coxeter element, one knows, by a result of Coleman and Coxeter (See [1] and [5], Section 8.3.)

PROPOSITION 3.4.2. *The eigenvalues of τ in h are the numbers ω^{m_i}, $i = 1, 2,..., l$.*

But now one can prove

LEMMA 3.4.2. *One has*

$$\prod_{\beta\in\Sigma} (1 - e^\beta)(a_Q) = h^l\, |\operatorname{Cent} K|.$$

Proof. Now a_Q is a principal element in K and, as noted in the proof of Proposition 3.4.1. Ad a_Q is a principal element of Ad $\mathbf{g}$. Since a_Q is a regular element in T the eigenvalues of Ad a_Q not equal to 1 are all the numbers of the form $e^\beta(a_Q)$ where $\beta \in \Sigma$. But now by Theorem 6.8 and Corollary 8.7 (see [5, bottom of p. 1026]) these numbers are exactly ω^k, $k = 1, 2,..., h - 1$, (See Section 8.3 in [5] which recalls that $q = O(\psi)$), each occurring with multiplicity l and also the numbers ω^{m_i}, $i = 1, 2,..., l$ each occurring once. Thus

$$\prod_{\beta\in\Sigma} (1 - e^\beta)(a_Q) = \det(1 - \tau) \left(\prod_{j=1}^{k-1} (1 - \omega^j)\right)^l.$$

But the result follows since $\det(1-\tau) = |\operatorname{Cent} K|$ by Lemma 3.4.1, and one knows that, for any positive integer n, $\prod_{j=1}^{n-1}(1-(e^{2\pi i/n})^j) = n$. Indeed this last statement is a special case of Lemma 3.4.1 where we choose $K = SU(n)$. Q.E.D.

3.5. Now recall Weyl's well known identity (of functions on T)

$$\prod_{\beta\in\Sigma+}(1-e^{-\beta}) = \sum_{\sigma\in W} sg\sigma e^{\sigma\rho-\rho}. \tag{3.5.1}$$

This is of course the starting point in [7]. This function is not Weyl group invariant. We now observe that if we sum this function over a Weyl group orbit we get a similar expression as before except that Σ replaces Σ_+.

Lemma 3.5.1. *Let $a \in T$. Then*

$$\sum_{\sigma\in W}\left(\prod_{\beta\in\Sigma+}(1-e^{-\beta})\right)(\sigma a) = \prod_{\beta\in\Sigma}(1-e^{\beta})(a).$$

Proof. Let $\xi = \sum_{\sigma\in W} sg\sigma e^{\sigma\rho}$ and let η be the function in (3.5.1). Thus $\xi e^{-\rho} = \eta$. Now the left side L of the equation in Lemma 3.5.1 equals (substituting σ^{-1} for σ) $\sum_{\sigma\in W}\eta(\sigma^{-1}a) = \sum_{\sigma\in W}\xi(\sigma^{-1}a)\, e^{-\rho}(\sigma^{-1}a)$. But $\xi(\sigma^{-1}a) = sg\sigma\xi(a)$ and $e^{-\rho}(\sigma^{-1}a) = e^{-\sigma\rho}(a)$. Thus $L = \xi(a)\sum_{\sigma\in W} sg\sigma e^{-\sigma\rho}(a) = (\xi\bar{\xi})(a)$ where bar denotes complex conjugation. But then we can write $L = (\xi e^{-\rho})(\overline{\xi e^{-\rho}})(a) = \eta\bar{\eta}(a) = \prod_{\beta\in\Sigma}(1-e^{\beta})(a)$. Q.E.D.

Now as in Section 3.3 let R denote either P or Q. We will now prove that the regular element $a_R \in T$, in some sense, "sees" the finite abelian group Γ_R in which it, up to W-conjugacy is the only regular element (Lemma 3.3).

Theorem 3.5. *One has*

$$\prod_{\beta\in\Sigma}(1-e^{\beta})(a_R) = |\Gamma_R|.$$

Proof. Since N and $M_R \subseteq N$ are stable under the action of W it follows that W operates on the quotient group N/M_R. Let $\hat{\rho} \in N/M_R$ be the image of ρ under coset projection and let $W_R = \{\sigma \in W \mid \sigma\hat{\rho} = \hat{\rho}\}$ so that W_R is a subgroup of W which may also be given by

$$W_R = \{\sigma \in W \mid \sigma\rho - \rho \in M_R\}. \tag{3.5.2}$$

We now integrate characters over the finite group Γ_R. If $\nu \in N$ then $e^\nu \mid \Gamma_R$ is the trivial character on Γ_R if and only if $\nu \in M_R$. Thus one has

$$\sum_{a \in \Gamma_R} e^\nu(a) = \begin{cases} 0 & \text{if } \nu \notin M_R \\ |\Gamma_R| & \text{if } \nu \in M_R. \end{cases} \tag{3.5.3}$$

We apply this formula for the case where $\nu = \sigma\rho - \rho$, multiply by $sg\sigma$ and sum over W. As a consequence of (3.5.2) and (3.5.3) one has

$$\sum_{a \in \Gamma_R} \Big(\sum_{\sigma \in W} sg\sigma e^{\sigma\rho - \rho}\Big)(a) = \Big(\sum_{\sigma \in W_R} sg\sigma\Big) |\Gamma_R|. \tag{3.5.4}$$

But now by (3.5.1) we may substitute $\prod_{\beta \in \Sigma+} (1 - e^{-\beta})$ for $\sum_{\sigma \in W} sg\sigma e^{\sigma\rho - \rho}$ in (3.5.4). But now $\prod_{\beta \in \Sigma+} (1 - e^{-\beta})(a) = 0$ if $a \in \Gamma$ is not regular. Thus by Lemma 3.3 we need not sum over all of Γ_R in (3.5.4) but only over the orbit Wa_R. But then by Lemma 3.5.1 we obtain the formula

$$\prod_{\beta \in \Sigma} (1 - e^\beta)(a_R) = \Big(\sum_{\sigma \in W_R} sg\sigma\Big) |\Gamma_R|.$$

But now the left side of this equation is not zero since a_R is regular. Thus $\sum_{\sigma \in W_R} sg\sigma \neq 0$ which implies that $\sigma \mapsto sg\sigma$ is the trivial character on W_R and hence one has

$$\prod_{\beta \in \Sigma} (1 - e^\beta)(a_R) = |W_R| |\Gamma_R|. \tag{3.5.5}$$

It suffices then to prove that W_R reduces to the identity. We first consider the case where $R = P$.

Let $Y \subseteq \mathbf{t}$ be the lattice defined by putting $Y = \{x \in \mathbf{t} \mid \exp x = e\}$ and let Y_P be the somewhat larger lattice defined by putting $Y_P = \{x \in \mathbf{t} \mid \exp x \in \Gamma_P\}$. It is clear that both Y and Y_P are stable under the action of W and hence exponentiation defines a W-isomorphism

$$f\colon Y_P/Y \to \Gamma_P. \tag{3.5.6}$$

Now for each $\nu \in \mathbf{h}^*$ let $x(\nu) \in \mathbf{h}$ be defined so that $(\mu, \nu) = \langle \mu, x(\nu) \rangle$ for all $\mu \in \mathbf{h}^*$. Now one knows that N is the $\mathbb{Z}$-span of all $\nu \in \mathbf{h}^*$ such that $(2(\nu, \beta)/(\beta, \beta)) \in \mathbb{Z}$ for all $\beta \in \Sigma$. Since Y is the dual lattice to N it is clear that Y is the $\mathbb{Z}$-span of the set $\{2\pi i(2x(\beta)/(\beta, \beta)) \mid \beta \in \Sigma\}$. It follows therefore that if $\gamma\colon \mathbf{t} \to \mathbf{h}^*$ is the W-isomorphism defined by the relation

$$(\mu, \gamma(y)) = \frac{1}{4\pi i} \langle \mu, y \rangle$$

for all $\mu \in \mathbf{h}^*$, so that if $\nu \in \mathbf{h}^*$,

$$\gamma(4\pi i x(\nu)) = \nu \tag{3.5.7}$$

then $\gamma(4\pi i(x(\beta)/(\beta, \beta))) = \beta/(\beta, \beta)$ for $\beta \in \Sigma$ and hence

$$\gamma(Y) = M_P. \tag{3.5.8}$$

We recall here that M_P is the $\mathbb{Z}$-span of the set $\{(\beta/(\beta, \beta)) \mid \beta \in \Sigma\}$. But now clearly $Y_P = \{y \in \mathbf{t} \mid (\langle \beta, y \rangle/(\beta, \beta)) \in 2\pi i\, \mathbb{Z}$ for all $\beta \in \Sigma\}$. If we put any $y \in \mathbf{t}$ in the form $y = 4\pi i\, x(\nu)$, $\nu \in \mathbf{h}^*$, so that $\gamma(y) = \nu$ the condition that $y \in Y_P$ is exactly the condition that $(2(\beta, \nu)/(\beta, \beta)) \in \mathbb{Z}$ for all $\beta \in \Sigma$. But this means that $\nu \in N$. Thus $\gamma(Y_P) = N$ and hence recalling (3.5.8) the map γ induces a W-isomorphism $\gamma\colon Y_P/Y \to N/M_P$. Thus if $\eta = \gamma \circ f^{-1}$ then

$$\eta\colon \Gamma_P \to N/M_P \tag{3.5.9}$$

is a W-isomorphism where for any $\nu \in N$

$$\eta(\exp 4\pi i x(\nu)) = \nu \bmod M_P. \tag{3.5.10}$$

But now recalling Section 3.1 one clearly has $4\pi i\, x(\rho) = x_P$. Thus $\eta(a_P) = \rho \bmod M_P = \hat{\rho}$. But now since a_P is regular (see Lemma 3.3) and since η is a W-isomorphism it follows from Proposition 3.2.3 that W_P reduces to the identity.

Recalling (3.5.5) to prove that W_Q reduces to the identity it is enough to directly show that $|\Gamma_Q| = h^l |\text{Cent } K|$ by Lemma 3.4.2. But now the isomorphism $N \to \hat{T}$, $\nu \mapsto e^\nu$ induces an isomorphism $N/M_Q \to \hat{\Gamma}_Q$ where $\hat{\Gamma}_Q$ is the character group of Γ_Q. Thus

$$|N/M_Q| = |\Gamma_Q|. \tag{3.5.11}$$

However $M_Q \subseteq L(\Sigma) \subseteq N$ so that $|N/M_Q| = |N/L(\Sigma)|\,|L(\Sigma)/M_Q|$. But since $M_Q = hL(\Sigma)$ clearly $|L(\Sigma)/M_Q| = h^l$. But $|N/L(\Sigma)| = |\text{Cent } K|$. (See (3.4.4)). Q.E.D.

Remark 3.5. In the course of the above proof we have shown that $W_R = \{e\}$. That is, if $\sigma \in W$ and $\sigma\rho - \rho \in M_R$ then $\sigma = e$. This statement is in fact contained in Macdonald's formula by considering in Theorem 2.4 the coefficient of $x^{(\rho,\rho)}$. The argument above provides an independent proof. We will draw some consequences of this fact.

LEMMA 3.5.2. *Let $\lambda \in D$ be arbitrary. Then either*

(1) *For all $\sigma \in W$, $\sigma(\lambda + \rho) - \rho \notin M_R$ or*

(2) *There exists a unique $\sigma \in W$ such that $\sigma(\lambda + \rho) - \rho \in M_R$.*

Proof. Assume that $\sigma_i^{-1}(\lambda + \rho) - \rho \in M_R$ for $\sigma_i \in W$, $i = 1, 2$. Then since M_R is W-stable one has $(\lambda + \rho) - \sigma_i\rho \in M_R$. But then $(\lambda + \rho - \sigma_2\rho) - (\lambda + \rho - \sigma_1\rho) = \sigma_1\rho - \sigma_2\rho \in M_R$. But if $\sigma = \sigma_2^{-1}\sigma_1$ then one also has $\sigma\rho - \rho \in M_R$. By Remark 3.5 this implies $\sigma = e$ so that $\sigma_1 = \sigma_2$. Q.E.D.

Now (for $R = P$ or Q) let ϵ_R be the function on D defined so that $\epsilon_R(\lambda) = 0$ in case $\sigma(\rho + \lambda) - \rho \notin M_R$ for all $\sigma \in W$ and otherwise $\epsilon_R(\lambda) = sg\sigma$ where $\sigma \in W$ is the unique element by (Lemma 3.5.2) such that $\sigma(\lambda + \rho) - \rho \in M_R$. Thus for any $\lambda \in D$

$$\epsilon_R(\lambda) = 1, 0, \text{ or } -1 \tag{3.5.12}$$

The relation of ϵ_R to Macdonald's formula is given in

LEMMA 3.5.3. *One has the following identity of formal power series*

$$\sum_{\nu \in M_R} d(\nu)\, x^{(\nu+\rho,\nu+\rho)} = \sum_{\lambda \in D} \epsilon_R(\lambda) \dim V_\lambda x^{(\lambda+\rho,\lambda+\rho)}. \tag{3.5.13}$$

Proof. An element $\mu \in N$ is called regular in case $(\mu, \beta) \neq 0$ for all $\beta \in \Sigma$. Let $M_R{}^0 = \{\nu \in M_R \mid \nu + \rho \text{ is regular}\}$. Also let $D^0 = \{\lambda \in D \mid \epsilon_R(\lambda) \neq 0\}$. Now if $\nu \in M_R$ then clearly $d(\nu) = 0$ if $\nu \notin M_R{}^0$. Thus we can substitute $M_R{}^0$ for M_R in the sum on the left hand side of (3.5.13) without changing its value. Similarly we can substitute D^0 for D on the right hand side of (3.5.13). But if $\nu \in M_R{}^0$ then $\nu + \rho$ is regular and there exists a unique $\sigma \in W$ such that $\sigma^{-1}(\nu + \rho) \in D$. But $\sigma^{-1}(\nu + \rho)$ is still regular and hence $\sigma^{-1}(\nu + \rho) - \rho = \lambda \in D$. Let $\delta\colon M_R{}^0 \to D$ be defined by putting $\delta(\nu) = \lambda$. But now clearly $\sigma(\lambda + \rho) - \rho = \nu \in M_R$. Thus $\lambda \in D^0$ by definition of ϵ_R and hence

$$\delta\colon M_R{}^0 \to D^0. \tag{3.5.14}$$

Also since $\sigma(\lambda + \rho) = \nu + \rho$ it follows that $d(\nu) = sg\sigma \dim V_\lambda$. Thus it suffices only to show that (3.5.14) is a bijective map. By (2) in Lemma 3.5.2 it is clear that it is injective. But if $\lambda \in D^0$ there exists $\sigma \in W$ such that $\nu = \sigma(\lambda + \rho) - \rho \in M_R$. However since $\nu + \rho = \sigma(\lambda + \rho)$ it follows that $\nu \in M_R{}^0$ since $\lambda + \rho$ is regular. Thus (3.5.14) is surjective. Q.E.D.

3.6. *Proof of Theorem* 3.1. Recall that, as in Section 3.3, R denotes either P or Q. By Theorem 2.4, Lemmas 3.5.3 and (3.5.12), to prove Theorem 3.1 it suffices only to prove

LEMMA 3.6. *For any $\lambda \in D$ one has*

$$\epsilon_R(\lambda) = \chi_\lambda(a_R)$$

where χ_λ is the character of π_λ.

Proof. By Weyl's character formula one has for any $\lambda \in D$,

$$\chi_\lambda \prod_{\beta \in \Sigma_+} (1 - e^{-\beta}) = \sum_{\sigma \in W} sg\sigma e^{\sigma(\lambda+\rho)-\rho}. \tag{3.6.1}$$

We will integrate both sides of (3.6.1) over the finite abelian group Γ_R. Considering first the right side of (3.6.1) we observe that

$$\sum_{a \in \Gamma_R} \left(\sum_{\sigma \in W} sg\sigma e^{\sigma(\lambda+\rho)-\rho} \right)(a) = \epsilon_R(\lambda) \mid \Gamma_R \mid. \tag{3.6.2}$$

Indeed if $\sigma(\lambda + \rho) - \rho \notin M_R$ then $e^{\sigma(\lambda+\rho)-\rho} \mid \Gamma_R$ is a non-trivial character of Γ_R and hence $\sum_{a\in\Gamma_R} e^{\sigma(\lambda+\rho)-\rho}(a) = 0$. But then if $\epsilon_R(\lambda) = 0$ the left side of (3.6.2) vanishes establishing (3.6.2) in this case. If $\epsilon_R(\lambda) \neq 0$ and $\sigma_0 \in W$ is the unique (by Lemma 3.5.2) element such that $\sigma_0(\lambda + \rho) - \rho \in M_R$ then the summand corresponding to any $\sigma \in W$ vanishes except for $\sigma = \sigma_0$. Furthermore $sg\sigma_0 = \epsilon_R(\lambda)$ and $\sum_{a\in\Gamma_R} e^{\sigma_0(\lambda+\rho)-\rho}(a) = \mid \Gamma_R \mid$. This proves (3.6.2) for all $\lambda \in D$.

Now with respect to the left side of (3.6.1) we first note that $(\chi_\lambda \prod_{\beta\in\Sigma+} (1 - e^{-\beta}))(a) = 0$ if $a \in \Gamma_R$ is not regular. Indeed $(1 - e^{-\beta})(a) = 0$ for some $\beta \in \Sigma_+$. Thus the integration on the left side of (3.6.1) is over the regular elements in Γ_R. But by Lemma 3.3 this means over the Weyl group orbit Wa_R.

Thus

$$\sum_{\sigma \in W} \left(\chi_\lambda(\sigma a_R) \prod_{\beta \in \Sigma_+} (1 - e^{-\beta})(\sigma a_R) \right) = \epsilon_R(\lambda) \mid \Gamma_R \mid. \tag{3.6.3}$$

But χ_λ is a class function and hence, for any $\sigma \in W$, $\chi_\lambda(\sigma a_R) = \chi_\lambda(a_R)$ now factors out on the left side of (3.6.3). But, by Lemma 3.5.1,

$$\sum_{\sigma\in W}\Big(\prod_{\beta\in\Sigma_+}(1-e^{-\beta})(\sigma a_R)\Big) = \prod_{\beta\in\Sigma}(1-e^{\beta})(a_R). \tag{3.6.4}$$

On the other hand by Theorem 3.5 the right side of (3.6.4) equals $|\Gamma_R|$. Thus (3.6.3) reduces to the equation $\chi_\lambda(a_R)\,|\Gamma_R| = \epsilon_R(\lambda)\,|\Gamma_R|$. This proves Lemma 3.6 and hence also Theorem 3.1. Q.E.D.

4. The Action of the Coxeter Element on a Zero Weight Space

4.1. If V is any K-module then, clearly, one obtains a natural representation of the Weyl group $W = N(T)/T$ on the subspace V^T of T-invariants in V. (The usual action of W on $\mathbf{h}$ is of this form since $\mathbf{h} = \mathbf{g}^T$ with respect to Ad.) In fact for any $\lambda \in D$ let θ_λ denote the corresponding representation of W,

$$\theta_\lambda\colon W \to \operatorname{Aut} V_\lambda^T \tag{4.1.1}$$

on the zero weight space V_λ^T of V_λ. Thus θ_ψ, recalling that (see Section 2.2) ψ is the highest root, is the usual representation of W on $\mathbf{h}$.

Remark 4.1. In general θ_λ is not irreducible. However if $K = SU(n)$ there exists $\lambda_i \in D$, $1 = 1, 2,\ldots, p(n)$ (where $k \mapsto p(k)$ is the classical partition function) such that θ_{λ_i} is irreducible and θ_{λ_i}, $i = 1, 2,\ldots, p(n)$ runs through all the irreducible representations of W. Recall that W is isomorphic to the symmetric group S_n. The λ_i are determined as follows: One knows that there is a natural representation (as in [11]) of $SU(n)$ on the r-fold tensor product $\otimes^r \mathbb{C}^n$ for any $r \in \mathbb{Z}_+$. However, one shows easily that $(\otimes^r \mathbb{C}^n)^T = 0$ unless r is a multiple of n. In particular one easily has $\dim(\otimes^n \mathbb{C}^n)^T = n!$ On the other hand from the general theory of the decomposition of tensors into symmetry types one knows (see [11]) that there exists $\lambda_i \in D$, $t = 1, 2,\ldots, p(n)$, distinct, and multiplicities d_i, $i = 1,\ldots, p(n)$ such that as an $SU(n)$-module, $\otimes^n \mathbb{C}^n = \sum d_i V_{\lambda_i}$. This gives the primary decomposition $\otimes^n \mathbb{C}^n$ as an $SU(n)$-module. On the other hand the same theory shows very easily that $d_i = \dim V_{\lambda_i}^T$, that the action of W on $(\otimes^n \mathbb{C}^n)^T$ is equivalent to the left regular representation of W and that in fact, as W-modules,

$$(\bigotimes^n \mathbb{C}^n)^T \cong \sum d_i V_{\lambda_i}^T \tag{4.1.2}$$

is the primary decomposition of $(\bigotimes^n \mathbb{C}^n)^T$. In particular the θ_{λ_i} are irreducible and run through all irreducible representations of W.

Let $\tau \in W$ be a Coxeter element. The following theorem relates the operator $\theta_\lambda(\tau)$, $\lambda \in D$, to Macdonald's formulas. Now if one looks at a character table for the symmetric group S_n it is striking (so it seemed to me), that the character values for the permutation with one cycle was always 1, 0 or -1. (A proof of this fact was shown to me by G. Lustig). In the notation of Remark 4.1 this means that if $K = SU(n)$ then $\operatorname{tr} \theta_{\lambda_i}(\tau) = 1$, 0 or -1 for $i = 1, 2, \ldots, p(n)$ where τ is a Coxeter element. The following theorem contains a generalization ((4.1.2)) of this fact. On the other hand the Jacobi formula for $\varphi(X)^3$ (see (1.1.1)) is the case where $K = SU(2)$ and the coefficient of $\dim V_\lambda x^{c(\lambda)}$ (see (2.2.2)) is clearly recognizable to be $\operatorname{tr} \theta_\lambda(\tau)$. (It was this fact which suggested the main result of this paper to me). The following theorem also contains a direct generalization ((4.1.4)) of the Jacobi formula (1.1.1).

THEOREM 4.1. *Let $\varphi(x)$ be the power series* (2.3.1). *Also let $\tau \in W$ be a Coxeter element and for any $\lambda \in D$ let θ_λ be the representation* (4.1.1), *of the Weyl group W on the zero weight space $V_\lambda{}^T$. Then one has*

$$\operatorname{tr} \theta_\lambda(\tau) = 1, 0 \text{ or } -1 \tag{4.1.2}$$

and one also has

$$\left(\prod_{i=1}^{l} \varphi(x^{h(\alpha_i, \alpha_i)})\right)^{h+1} = \sum_{\lambda \in D} \operatorname{tr} \theta_\lambda(\tau) \dim V_\lambda\, x^{c(\lambda)} \tag{4.1.3}$$

where $c(\lambda)$ is the eigenvalue of the Casimir element in V_λ. (*See Section* 2.2). *In particular if K is simply-laced*

$$\varphi(s)^d = \sum_{\lambda \in D} \operatorname{tr} \theta_\lambda(\tau) \dim V_\lambda\, x^{c(\lambda)} \tag{4.1.4}$$

where $d = \dim K$.

Proof. We first observe that (4.1.3) does reduce to (4.1.4) in case K is simply-laced. Indeed this follows since, in the simply-laced case,

$h(\alpha_i, \alpha_i) = 1$ (Proposition 2.2) and $l(h+1) = d$. (See e.g. Section 8.3 and Theorem 8.4 in [5]).

But now by 8.12 in [7] one has the formula

$$(\rho, \rho) = \frac{h(h+1)}{24} \sum_{1=1}^{l} (\alpha_i, \alpha_i)$$

it follows therefore that

$$\left(\prod_{i=1}^{l} \varphi(x^{h(\alpha_i,\alpha_i)})\right)^{h+1} = \left(\prod_{i=1}^{l} \eta(x^{h(\alpha_i,\alpha_i)})\right)^{h+1} x^{-(\rho,\rho)}.$$

But then by (2.2.2) and the second formula in Theorem 3.1 the left hand side of (4.1.3) equals $\sum_{\lambda \in D} \chi_\lambda(a_Q) \dim V_\lambda x^{c(\lambda)}$. It suffices therefore by Theorem 3.1 to prove

$$\chi_\lambda(a_Q) = \operatorname{tr} \theta_\lambda(\tau). \tag{4.1.5}$$

But now for any weight μ of π_λ let $V_\lambda(\mu) \subseteq V_\lambda$ be the corresponding weight space. But by Proposition 3.4.1 it follows that if $a \in N(T)$ is such that $aT = \tau$ then a is a principal element of K. Thus a is conjugate to a_Q and hence

$$\chi_\lambda(a) = \chi_\lambda(a_Q).$$

But clearly $\pi_\lambda(a) V_\lambda(\mu) \subseteq V_\lambda(\tau\mu)$. On the other hand if $\mu \neq 0$ one knows that $\tau\mu \neq \mu$ since by a result of Coleman 1 is not an eigenvalue of τ in $\mathbf{h}^*$. (See e.g. Lemma 8.1 in [5]).

Thus $\chi_\lambda(a) = \operatorname{tr} \pi_\lambda(a) \mid V_\lambda(0) = \operatorname{tr} \theta_\lambda(\tau)$. This proves (4.1.5). Q.E.D.

4.2. Now let S be the algebra of polynomial functions on $\mathbf{g}$. Then S is a (locally finite) K-module where if $f \in S$, $x \in \mathbf{g}$ and $a \in K$ then $(a \cdot f)(x) = f(\operatorname{Ad} a^{-1}x)$. Also S is a graded algebra. Its homogeneous subspace of degree n will be denoted by S_n. Let $J = S^K$ be the algebra of K-invariant polynomials in S. Then one knows (Chevalley) that $J = \mathbb{C}[z_1, \ldots, z_l]$ is a polynomial ring in l algebraically independent generators $z_1, \ldots, z_l$. Furthermore the z_i can be chosen to be homogeneous. In fact they may be chosen so that $z_i \in S_{m_i+1}$ where the m_i are the exponents of K. See Section 3.4.

Now let J_* be the set of all differential operators ∂ on $\mathbf{g}$ with constant coefficients such that (1) $\partial 1 = 0$ and (2) ∂ commutes with the action of K on S. A polynomial $f \in S$ is called harmonic if $\partial f = 0$ for all

$\partial \in J_*$. Let $H \subseteq S$ denote the subspace of all harmonic polynomials. It is clear that H is a graded subspace of S. One knows that (see Theorem 11, [6])

$$S = J \otimes H \tag{4.2.1}$$

where tensor product corresponds to multiplication. Clearly H is a K-submodule of S. Let

$$H = \bigoplus_{\lambda \in D} H_\lambda \tag{4.2.2}$$

be the primary decomposition of H where H_λ, $\lambda \in D$, denotes the primary component corresponding to the representation π_λ. Not only is H_λ finite dimensional but more precisely one knows (see Theorem 11, [6]) that the multiplicity of π_λ in H equals $l(\lambda)$ where $l(\lambda) = \dim V_\lambda{}^T$. Thus one has a further reduction

$$H_\lambda = \bigoplus_{i=1}^{l(\lambda)} H_\lambda{}^i \tag{4.2.3}$$

where the $H_\lambda{}^i$ are K-irreducible. This decomposition is not unique. However there clearly exists a unique sequence of integers $m_1(\lambda) \leqslant \cdots \leqslant m_{l(\lambda)}(\lambda)$ so that the $H_\lambda{}^i$ can be chosen to satisfy

$$H_\lambda{}^i \subseteq H_{m_i(\lambda)} \tag{4.2.4}$$

where $H_n = H \cap S_n$.

Remark 4.2. The integer $m_i(\lambda)$ are called generalized exponents. The terminology is justified because if $\psi \in \Sigma$ is the highest root then $l(\psi) = l$ and the $m_i(\psi)$ are the usual exponents. See [6, Remark 26, p. 397].

Now if $\tau \in W$ is a Coxeter element then generalizing the Coleman–Coxeter result (see Proposition 3.4.2) that the eigenvalues of $\theta_\psi(\tau)$ are ω^{m_i}, $i = 1, 2, \ldots, l$, where $\omega = e^{2\pi i/h}$, we have proved

PROPOSITION 4.2.1. *For any $\lambda \in D$ the eigenvalues of $\theta_\lambda(\tau)$ are $\omega^{m_i(\lambda)}$, $i = 1, 2, \ldots, l(\lambda)$. In particular then*

$$\operatorname{tr} \theta_\lambda(\tau) = \omega^{m_1(\lambda)} + \cdots + \omega^{m_{l(\lambda)}(\lambda)}. \tag{4.2.2}$$

(Both sides are interpreted to be zero in case $l(\lambda) = 0$).

Proof. See Theorem 19 in [6]. Q.E.D.

The following result will have a simpler form (see Theorem 5.4) after we introduce the Mellin transform.

THEOREM 4.2. *For any $\lambda \in D$ one has*

$$\omega^{m_1(\lambda)} + \cdots + \omega^{m_{l(\lambda)}(\lambda)} = 1, 0 \text{ or } -1 \tag{4.2.3}$$

where $\omega = e^{2\pi i/h}$ and the $m_i(\lambda)$ are generalized exponents. Moreover

$$\left(\prod_{i=1}^{l} \eta(x^{h(\alpha_i,\alpha_i)})\right)^{h+1} = \sum_{\lambda\in D} \sum_{i=1}^{l(\lambda)} \omega^{m_i(\lambda)} \dim V_\lambda x^{(\lambda+\rho,\lambda+\rho)}.$$

In particular if K is simply-laced

$$\eta(x)^d = \sum_{\lambda\in D} \sum_{i=1}^{l(\lambda)} \omega^{m_i(\lambda)} \dim V_\lambda x^{(\lambda+\rho,\lambda+\rho)}. \tag{4.2.4}$$

Proof. This follows immediately from (4.2.2), (4.1.5) Theorem 3.1 and Proposition 3.1. Q.E.D.

5. THE ZETA FUNCTION FOR K EVALUATED AT a_p

5.1. We will recall certain well known facts concerning the relation between modular forms and Dirichlet series. Our main reference here is [8], Introduction and Chapter I. Let $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half-plane. Let $\gamma > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\gamma} \tag{5.1.1}$$

is a holomorphic function on $\mathbb{C}^+$ in case the complex numbers

$$a_n = O(n^c), \qquad n \to \infty \tag{5.1.2}$$

for some positive number c. One says that f is a modular form of weight $k > 0$ in case $f(-1/z) = (z/i)^k f(z)$ where $(z/i)^k = e^{k \log z/i}$ and log is real on the positive real axis.

If f, satisfying (5.1.1) and (5.1.2) is a modular form of weight k then (See [8, Theorem 1, Chapter I]) for Re s sufficiently large the Mellin transform

$$\Phi(s) = \int_0^{\infty} t^{s-1}(f(it) - a_0)\, dt \tag{5.1.3}$$

is an absolutely convergent integral. Moreover the Dirichlet series $\delta(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges to a holomorphic function $\delta(s)$, also for $\operatorname{Re} s$ sufficiently large. But in fact Φ and δ extend to meromorphic functions in the entire plane $\mathbb{C}$ and one has in $\mathbb{C}$

$$\Phi(s) = \left(\frac{2\pi}{\gamma}\right)^{-s} \Gamma(s)\,\delta(s) \tag{5.1.4}$$

where Γ is the usual gamma function.

Furthermore

$$\Phi(s) + \frac{a_0}{s} + \frac{a_0}{k-s} \qquad \text{is entire} \tag{5.1.5}$$

and bounded in every vertical strip. Finally Φ satisfies

$$\Phi(k-s) = \Phi(s) \tag{5.1.6}$$

for all $s \in \mathbb{C}$.

Remark 5.1.5. Note that if $a_0 = 0$ then Φ, itself, is entire by (5.1.5).

It is easy to see that if f, g satisfy (5.1.1), (5.1.2) (same γ, but possibly different c-values) and are, respectively, modular of weights k and l then fg satisfies (5.1.1) and (5.1.2) for some c and is modular of weight $k + l$. But now if $\eta(x)$ is the Dedekind η-function and we put $x = e^{2\pi i z}$ for $z \in \mathbb{C}_+$ then $\tilde{\eta}(z) = \eta(e^{2\pi i z})$ satisfies the conditions (5.1.1) and (5.1.2) where $\gamma = 1/24$. This follows, for example, from Euler's formula for $\varphi(x)$ (See [4], Theorem 3.5.3, p. 284). On the other hand $\tilde{\eta}$ is modular of weight $\frac{1}{2}$. (See e.g., [8, p. 1–43]). Thus $\tilde{\eta}^d$, $d = \dim K$, is modular of weight $d/2$. Thus if $M\eta^d(s)$ is the Mellin transform of $\tilde{\eta}^d$ evaluated at $s \in \mathbb{C}$ then $M\eta^d$ is entire (Since clearly, $a_0 = 0$ for $\tilde{\eta}$ and hence $\tilde{\eta}^d$. See Remark 5.1.5) and satisfies

$$M\eta^d(s) = M\eta^d\left(\frac{d}{2} - s\right). \tag{5.1.7}$$

5.2. Now let $L_2(K)$ be the Hilbert space of square integrable functions on K with respect to Haar measure. The left regular (unitary) representation $a \mapsto L_a$ of K on $L_2(K)$ is defined so that if $a, b \in K$, $f \in L_2(K)$ then $(L_a f)(b) = f(a^{-1}b)$.

For each $\lambda \in D$ let $E_\lambda \subseteq L_2(K)$ be the π_λ-primary component of $L_2(K)$ so that $\dim E_\lambda = (\dim V_\lambda)^2$ and if $E = \bigoplus_{\lambda \in D} E_\lambda$ then E is dense in $L_2(K)$ and is equal to the set of all K-finite smooth functions on K.

The space E is then in a natural way a module for $\mathbf{k}$ and the enveloping algebra U. It is easy to see that the closure, Δ, of the operator in E corresponding to the Casimir element (see Section 2.2) is just the Laplace–Beltrami operator on K defined with respect to the Riemannian metric on K corresponding to the negative of the Killing form. In particular then, by (2.2.2), E_λ is contained in the eigenspace for $\Delta + (\rho, \rho)$ belonging to the eigenvalue $(\lambda + \rho, \lambda + \rho)$. Thus if $s \in \mathbb{C}$ it follows that $(\Delta + (\rho, \rho))^{-s}$ reduces to the scalar $(\lambda + \rho, \lambda + \rho)^{-s}$ on E_λ. On the other hand one knows that

$$\sum_{\lambda \in D} (\dim V_\lambda)^2 (\lambda + \rho, \lambda + \rho)^{-s} < \infty \tag{5.2.1}$$

in case $\operatorname{Re} s > d/2$. See e.g. [10, Lemma 5.6.7, p. 126]. (It is clear that there exists a constant $c > 0$ such that $c(1 + (\lambda, \lambda)) \geqslant (\lambda + \rho, \lambda + \rho))$ for all $\lambda \in D$). Thus if $\operatorname{Re} s > d/2$ the operator $(\Delta + (\rho, \rho))^{-s}$ on $L_2(K)$ is of trace class. But then so is $(\Delta + (\rho, \rho))^{-s} L_a^{-1}$ for $a \in K$. One then obtains a continuous function on K, which we identify with $(\Delta + (\rho, \rho))^{-s}$ itself, by putting, for any $a \in K$,

$$(\Delta + (\rho, \rho))^{-s}(a) = \operatorname{tr}(\Delta + (\rho, \rho))^{-s} L_a^{-1}. \tag{5.2.2}$$

The identification should not cause confusion here since it is easy to see that the operator $(\Delta + (\rho, \rho))^{-s}$ is just left convolution by the function $(\Delta + (\rho, \rho))^{-s}$. The Fourier expansion of this function is given in

PROPOSITION 5.2. *Assume* $\operatorname{Re} s > d/2$. *Then for any* $a \in K$ *one has*

$$(\Delta + (\rho, \rho))^{-s}(a) = \sum_{\lambda \in D} \chi_\lambda(a^{-1}) \dim V_\lambda (\lambda + \rho, \lambda + \rho)^{-s}$$

and the sum converges absolutely.

Proof. The absolute convergence follows from (5.2.1) and the fact that $|\chi_\lambda(a)| \leqslant \dim V_\lambda$ for any $a \in K$. The equality follows from (5.2.2) and the fact that

$$\operatorname{tr}((\Delta + (\rho, \rho))^{-s} L_a^{-1} \mid E_\lambda) = \chi_\lambda(a^{-1}) \dim V_\lambda (\lambda + \rho, \lambda + \rho)^{-s}.$$

One of course uses the fact that π_λ occurs with multiplicity equal to $\dim V_\lambda$ in E_λ.

We now show that if we evaluate the function $(\Delta + (\rho, \rho))^{-s}$ at the regular element a_P the corresponding function in s extends to an entire function satisfying a functional equation. More precisely one has

THEOREM 5.2. *If a_P is the regular element in K defined in Section 3.1 one has*

$$M\eta^d(s) = (2\pi)^{-s}\,\Gamma(s)(\Delta + (\rho, \rho))^{-s}\,(a_P)$$

where $M\eta^d$ is the Mellin transform of the dth power ($d = \dim K$) of the Dedekind η-function.

Proof. We first note that there exists a rational number $\gamma > 0$ such that $(\lambda + \rho, \lambda + \rho)$ for any $\lambda \in D$ is of the form $(\lambda + \rho, \lambda + \rho) = n/\gamma$ for some integer $n \in \mathbb{Z}$. This follows, for example, by writing λ as an integral combination of the λ_i, defined after equation (3.4.4), and then noting that all inner products (λ_i, λ_j), (λ_i, ρ) are rational numbers. It follows that the absolutely convergent sum (5.2.1) can be resummed so that for $\operatorname{Re} s > d/2$

$$\sum_{\lambda \in D} (\dim V_\lambda)^2\,(\lambda + \rho, \lambda + \rho)^{-s} = \gamma^s \sum_{n=1}^{\infty} b_n n^{-s}.$$

But since these sums converge one must have $b_n = O(n^c)$ for some $c > 0$. On the other hand $|\,\chi_\lambda(a_P)| \leqslant \dim V_\lambda$ so that if a_n is defined by

$$\sum_{\lambda \in D} \chi_\lambda(a_P) \dim V_\lambda(\lambda + \rho, \lambda + \rho)^{-s} = \gamma^s \sum_{n=1}^{\infty} a_n n^{-s} \tag{5.2.3}$$

one clearly has $|\,a_n\,| \leqslant b_n$ so that $a_n = O(n^c)$ for some $c > 0$ and both sums in (5.2.3) are absolutely convergent for $\operatorname{Re} s > d/2$. The a_n are determined by the rearrangement of the left side of (5.2.3) so that if $x^{(\lambda+\rho,\lambda+\rho)}$, x indeterminate, is substituted for $(\lambda + \rho, \lambda + \rho)^{-s}$ on the left hand side of (5.2.3) one retains equality in (5.2.3) when $x^{n/\gamma}$ is substituted for $\gamma^{-s}n^{-s}$ on the right hand side of (5.2.3). On the other hand fixing s where $\operatorname{Re} s > d/2$ and $t > 0$ one clearly has $|(\lambda + \rho, \lambda + \rho)^{-s}\,| \geqslant e^{-2\pi t(\lambda+\rho,\lambda+\rho)}$ for $(\lambda + \rho, \lambda + \rho)$ sufficiently large. Thus one has ($x = e^{-2\pi t}$)

$$\sum_{\lambda \in D} \chi_\lambda(a_P) \dim V_\lambda e^{-2\pi t(\lambda+\rho,\lambda+\rho)} = \sum_{n=1}^{\infty} a_n e^{-2\pi t(n/\gamma)} \tag{5.2.4}$$

where both sums are absolutely convergent. But by Theorem 3.1

if $\tilde{\eta}$ is defined as in Section 5.1 one has $\tilde{\eta}^d(it)$ is exactly given by the left hand side of (5.2.4). But

$$M\eta^d(s) = \int_0^\infty t^{s-1}\tilde{\eta}^d(it)\,dt \tag{5.2.5}$$

for Re s sufficiently large. But the Mellin transform of a term on the left hand side of (5.2.4) is

$$\begin{aligned}\chi_\lambda(a_P)\dim V_\lambda \int_0^\infty t^{s-1}e^{-2\pi t(\lambda+\rho,\lambda+\rho)}\,dt \\ = (2\pi)^{-s}\,\Gamma(s)\,\chi_\lambda(a_P)\dim V_\lambda(\lambda+\rho,\lambda+\rho)^{-s}.\end{aligned} \tag{5.2.6}$$

However by Proposition 5.2 the sum of (5.2.6), over all $\lambda \in D$, absolutely converges to $2\pi^{-s}\Gamma(s)\ (\Delta + (\rho,\rho))(a_P)$ for Re $s > d/2$. We are using here the fact that a_P is conjugate to a_P^{-1}. This is clear from the definition of a_P and the fact that ρ is Weyl group conjugate to $-\rho$. Thus we are done as soon as we can justify the interchange of the sum with the integral. But since the sum of (5.2.6) is absolutely convergent we can recollect the terms so that, recalling (5.2.4), the sum of (5.2.6) is the same as the sum of the corresponding integrals of the terms on the right side of (5.2.4). But the interchange of sum and integral is well known in this case and is just the statement (5.1.4). Q.E.D.

5.3. Recall the notation of Section 4.2 and in particular the double sum (4.2.4). Let I denote the set of all pairs (λ, j) where $\lambda \in D$ and $1 \leqslant j \leqslant l(\lambda)$. We partition I into a disjoint union $I = \bigcup_{k=0}^{\infty} I_k$ where I_k is the finite subset defined by putting $I_k = \{(\lambda, j) \in I \mid m_j(\lambda) = k\}$. It is then clear from (4.2.2), (4.2.3) and (4.2.4) that

$$\binom{k+d-1}{d-1} = \dim S_k \geqslant \dim H_k = \sum_{(\lambda,j)\in I_k} \dim V_\lambda\,. \tag{5.3.1}$$

Lemma 5.3.1. *There exists a positive constant r such that for all $k \in \mathbb{Z}_+$ and any $(\lambda, j) \in I_k$*

$$(\lambda+\rho,\lambda+\rho) \geqslant (\rho,\rho) + kr.$$

Proof. For any $\nu \in \mathbf{h}^*$ let $O(\nu) = (h/2\pi i)\langle \nu, x_Q\rangle$ so that by (3.1.2) one has $O(\alpha_i) = 1$ for any simple root α_i. Let $\lambda \in D$. We have proved that if $l(\lambda) \neq 0$ (i.e. $H_\lambda \neq 0$) then $O(\lambda) = m_{l(\lambda)}(\lambda)$. See Theorem 17, p. 396 in [6]. It follows therefore that for any $(\lambda, j) \in I_k$ one has

$$O(\lambda) \geqslant k. \tag{5.3.2}$$

On the other hand since $2(\rho, \alpha_i)/(\alpha_i, \alpha_i) = 1$ for any simple root α_i it follows that if $r = \min_i(\alpha_i, \alpha_i)$ then $(2(\rho, \alpha_i)/r) \geqslant 1 = O(\alpha_i)$. But if $(\lambda, j) \in I_k$ then the coefficients of λ with respect to the α_i are positive (since $\lambda \in D$) and hence $(2(\rho, \lambda)/r) \geqslant O(\lambda) \geqslant k$. Thus $2(\rho, \lambda) \geqslant kr$ and thus certainly $(\lambda, \lambda) + 2(\rho, \lambda) + (\rho, \rho) = (\lambda + \rho, \lambda + \rho) \geqslant (\rho, \rho) + kr$. Q.E.D.

Let $\tilde{\eta}$ be as in Section 5.1.5. Recall $\omega = e^{2\pi i/h}$.

LEMMA 5.3.2. *Assume that K is simply-laced. For any $t > 0$ one has*

$$\tilde{\eta}(it) = \sum_{k=0}^{\infty} \sum_{(\lambda,j)\in I_j} \omega^{m_j(\lambda)} \dim V_\lambda e^{-2\pi t(\lambda+\rho,\lambda+\rho)} \tag{5.3.3}$$

and the sum is absolutely convergent.

Proof. By (5.3.1) and Lemma 5.3.1 one clearly has for any $k \in \mathbb{Z}_+$

$$\binom{k+d-1}{d-1} e^{-2\pi t((\rho,\rho)+kr)} \geqslant \sum_{(\lambda,j)\in I_k} | \omega^{m_j(\lambda)} \dim V_\lambda e^{-2\pi t(\lambda+\rho,\lambda+\rho)} |. \tag{5.3.4}$$

But the sum of the left side of (5.3.4) over $k \in \mathbb{Z}_+$ is exactly $e^{-2\pi t(\rho,\rho)}(1/(1 - e^{-2\pi tr})^d)$. Thus

$$e^{-2\pi t(\rho,\rho)} \left(\frac{1}{1 - e^{-2\pi \mathrm{tr}}}\right)^d \geqslant \sum_{k=0}^{\infty} \sum_{(\lambda,j)\in I_k} | \omega^{m_j(\lambda)} \dim V_\lambda e^{-2\pi t(\lambda+\rho,\lambda+\rho)} |. \tag{5.3.5}$$

This proves the absolute convergence of the right side of (5.3.3). But then by (4.2.4) the sum of the right side of (5.3.3) equals $\tilde{\eta}(it)$. Q.E.D.

5.4. If A is an operator on a countably dimensional vector space (over $\mathbb{C}$) V we will say that A is traceable and tr A is its trace if there exists a number, Tr A, such that with respect to any basis of V the sum of the diagonal terms of the corresponding matrix for A converges absolutely to tr A. If there exists a basis v_i of V consisting of eigenvectors for A such that $\sum_i | \lambda_i |$ converges where λ_i is the corresponding eigenvalues it is easy to see that A is traceable and tr $A = \sum_i \lambda_i$.

Now the space H of harmonic polynomials on $\mathbf{k}$ (see Section 4.2) is a K-module of finite type and hence is a module for the universal enveloping algebra U. Let $\Box$ be the operator on H corresponding to the Casimir element z. (See Section 2.2). Also let Ω be the operator on H defined so that Ω reduces to the scalar ω^j on H_j where as usual, $\omega = e^{2\pi i/h}$.

THEOREM 5.4. *If* $\operatorname{Re} s > d + 1$ *then* $\Omega(\Box + (\rho, \rho))^{-s}$ *is a traceable operator on H. Moreover if K is simply-laced then* $s \mapsto \operatorname{tr} \Omega(\Box + (\rho, \rho))^{-s}$ *extends to an entire function of* $s \in \mathbb{C}$ *and satisfies a functional equation. In fact*

$$M\eta^d(s) = (2\pi)^{-s}\,\Gamma(s)\operatorname{tr}\Omega(\Box + (\rho, \rho))^{-s}$$

where $M\eta^d$ *is the Mellin transform of the* dth ($d = \dim K$) *power of the Dedekind* η*-function.*

Proof. If $\operatorname{Re} s > d + 1$ note that the function

$$f(t) = e^{-2\pi t(\rho,\rho)} \left(\frac{1}{1 - e^{-2\pi t r}}\right)^d t^{s-1}$$

is integrable in the interval $[0, \infty)$. That is, it is bounded as $t \to 0$ and exponentially tends to zero as $t \to \infty$. It follows therefore from (5.3.4) that if $f_{\lambda,j}(t) = t^{s-1}\omega^{m_j(\lambda)} \dim V_\lambda e^{-2\pi t(\lambda+\rho,\lambda+\rho)}$ where $(\lambda, j) \in I$ then

$$\sum_{(\lambda,j)\in I} \int_0^\infty |f_{\lambda,j}(t)|\, dt$$

is finite. But then if K is simply-laced, by Lemma 5.3.2, for $\operatorname{Re} s > d + 1$ the function $t \mapsto t^{s-1}\tilde{\eta}^d(it)$ is integrable in $[0, \infty)$ and by 5.2.5 one has

$$M\eta^d(s) = \sum_{k=0}^{\infty} \sum_{(\lambda,j)\in I_k} \int_0^\infty f_{\lambda,j}(t)\, dt \tag{5.4.1}$$

where the sum is absolutely convergent. But

$$\int_0^\infty f_{\lambda,j}(t)\, dt = (2\pi)^{-s}\,\Gamma(s)\,\omega^{m_j(\lambda)} \dim V_\lambda(\lambda + \rho, \lambda + \rho)^{-s}. \tag{5.4.2}$$

On the other hand recalling (4.2.2), (4.2.3) and (4.2.4) one has

$$H_k = \bigoplus_{(\lambda, j) \in I_k} H_\lambda{}^j \quad \text{and} \quad H = \bigoplus_{k=0}^{\infty} H_k . \tag{5.4.3}$$

But $m_j(\lambda) = k$ for $(\lambda, j) \in I_k$ so that $\omega^{m_j(\lambda)}(\lambda + \rho, \lambda + \rho)^{-s}$ is just the eigenvalue of $\Omega(\Box + (\rho, \rho))^{-s}$ in the space $H_\lambda{}^j$. Thus $\omega^{m_j(\lambda)} \dim V_\lambda(\lambda + \rho, \lambda + \rho)^{-s} = \operatorname{tr} \Omega(\Box + (\rho, \rho))^{-s} \mid H_\lambda{}^j$. The absolutely convergent sum in (5.4.1) then implies by (5.4.3) that $\Omega(\Box + (\rho, \rho))^{-s}$ is traceable and $M\eta^d(s) = (2\pi)^{-s} \Gamma(s) \operatorname{tr} \Omega(\Box + (\rho, \rho))^{-s}$. The argument that the sum on the right side of (5.4.1) is absolutely convergent clearly does not require (5.3.3) so that $\Omega(\Box + (\rho, \rho))^{-s}$ is traceable whether or not K is simply-laced. Q.E.D.

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